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Continuum Modeling of the Mechanical and Thermal Behavior of Discrete Large Structures

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In the present paper we introduce a rather straightforward construction procedure in order to derive continuum equivalence of discrete truss-like repetitive structures. Once the actual structure is specified, the construction procedure can be outlined by the following three steps: 1) all sets of parallel members are identified, 2) unidirectional "effective continuum" properties are derived for each of these sets, and 3) orthogonal transformations are finally used to determine the contribution of each set to the "overall effective continuum" properties of the structure. Here the properties include mechanical (stiffnesses), thermal (coefficients of thermal expansions), and material densities. Once expanded descriptions of the steps 2 and 3 are done, the construction procedure will be applied to a wide variety of discrete structures and the results will be compared with those of other existing methods.

Nomenclature

A, A_1, A_2, A_d	= cross-sectional areas of rods
C_{ijkl}	= stiffness coefficients
$D^{(E)}_{ijkl}, D^{(T)}_{ij}, D^{(\rho)}$	= plate rigidities
E, E_1, E_2, E_d	= Young's modulii of rods
$F^{(E)}_{ijkl}, F^{(T)}_{ij}, F^{(\rho)}$	= plates first moments
h	= representative height
L, L_1, L_2, L_d	= member lengths
$Q^{(E)}_s, Q^{(T)}_s, Q^{(\rho)}_s$	= unidirectional properties
T	= change in absolute temperature T_0
$\alpha, \alpha_1, \alpha_2, \alpha_d$	= coefficients of linear thermal expansion
β_{ij}, β_p	= components of the orthogonal transformation tensor
γ_{ij}	= thermal expansion tensor
σ_{ij}	= stress tensor
ϵ_{ij}	= strain tensor
$\rho, \rho_1, \rho_2, \rho_d$	= material densities

I. Introduction

THE last decade has witnessed a dramatic increase in the research activities dealing with the possibility of utilizing space for various commercial and scientific needs. In several recent issues of *Astronautics & Aeronautics*[1-4] many articles have appeared which deal with diverse aspects of large space structures. These articles have identified various applications and also proposed novel designs of structures to meet such applications. A review of the research activities on space structures prior to 1975 has been documented in two volumes[5,6] that resulted from two successive International Conferences on Space Structures organized in 1966 and 1975, respectively.

It thus has become necessary to find and analyze small lightweight structures that will be used easily to construct much larger space structures. It would be desirable for these structures to be isotropic in nature. However, construction requirements may make this infeasible, therefore requiring orthotropic or possibly completely anisotropic structures. Truss-type periodic (repetitive) structures have recently been identified as candidates for space structures.[7,9] Here the possible simplicity in transportation and in space construction is most desirable.

In order to assess the utility of such discrete structures, complete understanding of their mechanical and thermal behavior is needed. A quick review of existing literature reveals, according to a widely quoted report,[10] three major possible techniques (see also Nemeth[11]). These consist of the direct methods, also known by the matrix structural analysis methods (e.g., Refs. 12-14); the discrete field methods, (e.g., Refs. 15-21); and, finally, the continuum methods (e.g., Refs. 22-29). Incidently, up to quite recently most of these methods were developed for analyzing on-ground discrete strutures.

In their purest sense the direct methods and the discrete field methods can be described as follows: The direct methods are characterized by writing the system of equations of equilibrium and compatability at each node and solving the resulting system to obtain nodal displacements and member forces. This of course applies to either repetitive or nonrepetitive structures. The discrete field methods, on the other hand, are attractive for repetitive structures where a representative unit cell of the structure is isolated and its equations of equilibrium are written in terms of differential (in time) finite difference (in space) format. Thus, these two methods as described above will not be practical in their applications to large space structures due to the large number of nodal points involved.

The continuum methods provide practical alternatives to the above methods, especially if one is interested in describing the overall behavior of repetitive structures; however, details of local (on the microscale) processes will not be predicted, herein. In its broadest sense the continuum approach is not unique. In fact, it borrows most of its fundamental ideas from the other two methods. For instance, in the case of repetitive structures, a representative repeating cell can be isolated and studied by the direct method. If one then relates its force or deformation characteristic (or both) to that of similar element of the continuum one can obtain the effective continuum properties (e.g., Refs. 25 and 27). A second distinct approach to the continuum modeling involves the expansion in Taylor series of the finite difference equations obtained using the discrete field methods (e.g., Renton[18]).

Another approach which has recently been widely utilized in developing substitute continuum to the discrete systems is the use of energy equivalence.[8,11,24,29] Essentially this amounts

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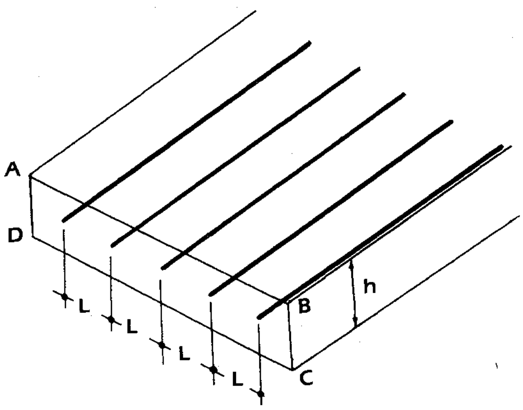

Fig. 1 Unidirectional-parallel members.

to equating the kinetic and strain energies of the repeating discrete cell to those of the similar continuum element. The nodal displacements and rotations of the cell are then expanded in a Taylor series to define the continuum models.

In the present paper we introduce a different and rather straightforward construction procedure in order to derive continuum equivalence of discrete truss-like repetitive structures. Once the actual structure is specified, the construction procedure can be outlined by the following three consecutive steps: 1) all sets of parallel members are identified, 2) unidirectional "effective continuum" properties are derived for each of these sets; and 3) orthogonal transformations are finally used to determine the contribution of each set to the "overall effective continuum" properties of the structure. Here the term properties is general and includes mechanical (stiffnesses), thermal (coefficients of thermal expansions), and material densities. Once expanded descriptions of the steps 2 and 3 are done, the construction procedure will be applied to a wide variety of discrete structures and the results will be compared with those of other existing methods.

II. Unidirectional-Effective Continuum Properties

A representative set of parallel members is shown in Fig. 1. In the context of truss-like structures it is assumed that each member of this set has the single unidirectional elastic property E, the single coefficient of linear thermal expansion α, and the mass density ρ. Due to the assumed periodicity, it is also assumed that the members are uniformly distributed (a distance L apart) in a rectangular array with an arbitrary height h as shown in the figure. (Here the height h is fictitiously introduced to introduce the extra dimension needed for the area average.) According to strength of materials analysis approaches the unidirectional effective continuum properties of the members of Fig. 1 are obtained by area-averaging the rod's properties. By examining Fig. 1, we see that each member effectively occupies the area Lh. Accordingly, the effective elastic property $Q^{(E)}$, the effective thermal expansion property $Q^{(T)}$, and the effective mass density $Q^{(\rho)}$ will be given respectively by

$$Q^{(E)}=\frac{EA}{Lh}, \quad Q^{(T)}=\frac{\alpha EA}{Lh}, \quad Q^{(\rho)}=\frac{\rho A}{Lh} \tag{1}$$

where A is the cross-sectional area of the individual rod. The results of Eqs. (1) are derived from basic principles of strength of material in which, under mechanical or thermal loadings, the deformation of the system of Fig. 1 is constrained by the requirement that all rod extensions or contractions are the same. The results displayed in Eqs. (1) can also be obtained as special cases of those corresponding to unidirectional fiber reinforced composites in which the properties of the matrix vanish (i.e., vacuum is simulated by materials with vanishing properties).

Remark

Although displayed in the most natural form, the results of Eqs. (1) are not unique. For instance, if the members of Fig. 1 are viewed as part of a larger structure then they might be required to occupy an effective area which is larger or smaller than the area of the rectangle ABCD. Under such a circumstance the values of Eqs. (1) will have to be area weighted again. For example, if these members are to occupy an area which is twice ABCD then their effective properties will be half of those listed in Eqs. (1).

III. Orthogonal Transformations

In this section the actual values of the total structure's effective continuum properties are determined from the individual contribution of each set of parallel members. The individual set's contribution is obtained by a three-dimensional coordinate transformation. Before we proceed to describe the transformation, however, we shall first review the relevant stress-strain-temperature relations of elastic bodies.

The thermoelastic stress-strain relations for a general linear elastic body are written in the compact form

$$\sigma_{ij}=C_{ijkl}\epsilon_{kl}+\gamma_{ij}T \qquad i,j,k,l=1,2,3 \tag{2}$$

where σ_{ij} and ϵ_{kl} are the components of the stress and strain tensors, respectively, T is the change in the absolute temperature T_0 and C_{ijkl} and γ_{ij} are the stiffnesses and thermal expansion tensors of the solid.

For future format reference we shall rewrite Eq. (2) in its expanded form

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}=\begin{bmatrix}C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112}\\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212}\\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312}\\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312}\\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312}\\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212}\end{bmatrix}$$

$$\times\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \epsilon_{23}\\ \epsilon_{13}\\ \epsilon_{12}\end{bmatrix}+\begin{bmatrix}\gamma_{11}\\ \gamma_{22}\\ \gamma_{33}\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix}T \tag{2'}$$

Since C_{ijkl}, γ_{ij}, and ρ are fourth-order, second-order, and zeroth-order tensors, respectively, they obey the transformation[30-32]

$$C_{ijkl}=C'_{pqrs}\beta_{pi}\beta_{qj}\beta_{rk}\beta_{sl} \tag{3}$$

$$\gamma_{ij}=\gamma'_{pq}\beta_{pi}\beta_{qj} \tag{4}$$

$$\rho(x_i)=\rho'(x'_i) \tag{5}$$

in which

$$\beta_{ij}=\partial x'_i/\partial x_j \tag{6}$$

are components of the orthogonal-transformation tensor which transforms the unprimed to the primed coordinates. Accordingly, β_{ij} is the cosine of the angle between the x'_i and the x_j axis.

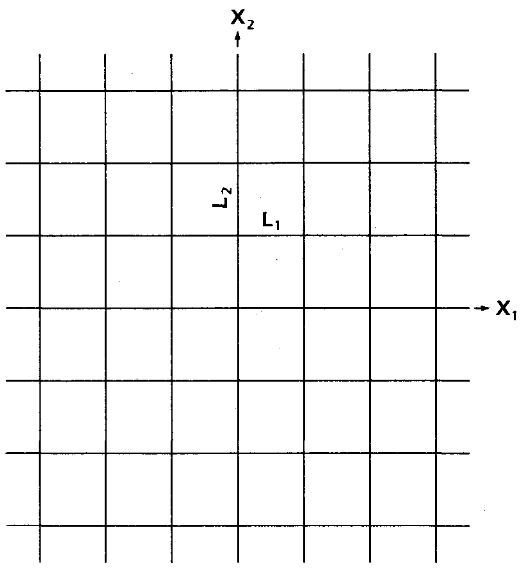

Fig. 2 (0°, 90°) layup.

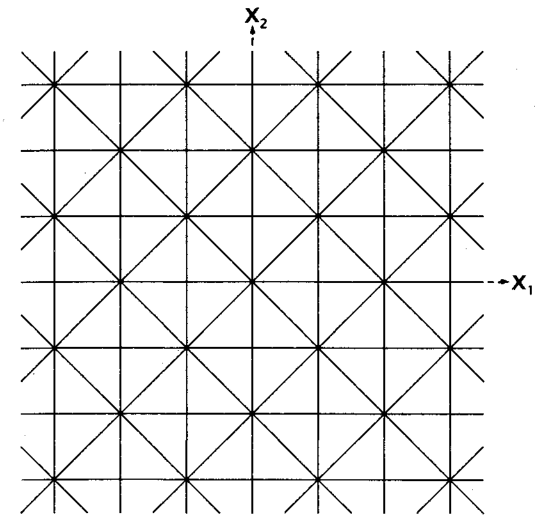

Fig. 3 (0°, 90°, ±45°) layup.

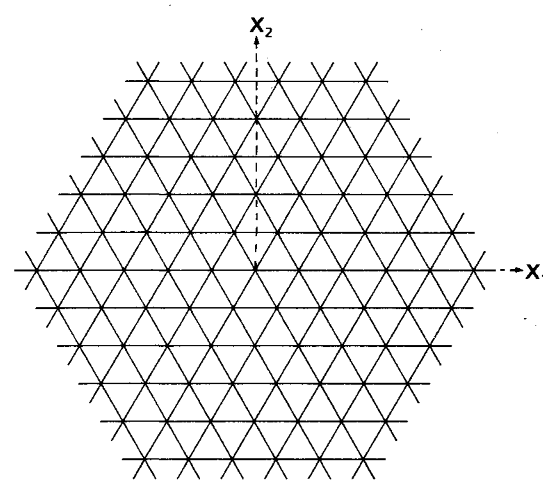

Fig. 4 (0°, ±60°) layup.

Equations (2-5) hold equally well for either continuous or discrete structures. The numerical values of the appropriate C_{ijkl} and γ_{ij} and ρ entries will depend, however, upon the specific structure under consideration. Since we are interested in analyzing truss-type structures that are constructed from unidirectional column elements, it is expected that each member will contribute to the overall properties. The sum of the average contribution of each member will then lead to the final properties.

Since unidirectional materials (like the present rods) have the single elastic property $C'_{1111}=E$, the single thermoelastic property $\gamma'_{11}=\alpha E$ and the material density $\rho'=\rho$, Eqs. (1) can be directly utilized to obtain the unidirectional effective continuum properties $Q^{(E)}$, $Q^{(T)}$, and $Q^{(\rho)}$ for each set of parallel members. Thus, if a structure has n different sets of parallel members then Eqs. (3-5) reduce for each set s, $s=,1,2,\ldots,n$ to

$$(C_{ijkl})_s = Q_s^{(E)}(\beta_{1i}\beta_{1j}\beta_{1k}\beta_{1l})_s \tag{7}$$

$$(\gamma_{ij})_s = Q_s^{(T)}(\beta_{1i}\beta_{1j})_s \tag{8}$$

$$(\rho)_s = Q_s^{(\rho)} \tag{9}$$

Furthermore, by identifying β_{1p} by β_p, $p=i,j,k,l$, we rewrite Eqs. (7-9) as

$$(C_{ijkl})_s = Q_s^{(E)}(\beta_i\beta_j\beta_k\beta_l)_s \tag{10a}$$

$$(\gamma_{ij})_s = Q_s^{(T)}(\beta_i\beta_j)_s \tag{10b}$$

$$(\rho)_s = Q_s^{(\rho)} \tag{10c}$$

Notice now that β_p, $p=1,2,3$ is the cosine of the angle between a specified set of parallel members and the x_p direction. In particular $(\beta_1,\beta_2,\beta_3)$ define the direction cosines of a given parallel members set with respect to the fixed coordinate system (x_1,x_2,x_3).

Once the direction cosines of each set of parallel members are identified, application of Eqs. (10) followed by the sum over all sets yield the final properties

$$C_{ijkl}=\sum_{s=1}^{n}(C_{ijkl})_s,\quad \gamma_{ij}=\sum_{s=1}^{n}(\gamma_{ij})_s,\quad \rho=\sum_{s=1}^{n}(\rho)_s \tag{11}$$

IV. Applications

In this section we present applications to our construction procedure as outlined in Secs. II and III. The models which we shall discuss constitute one-, two-, and three-dimensional truss-like structures, respectively.

One-Dimensional Structures

The assemblage of Fig. 1 is representative of one-dimensional structures. Assuming that the members are aligned along the x_1-direction, the effective properties of the structure will also be unidirectional, namely along the x_1-coordinate, and are constructed in accordance with Eqs. (1).

Two-Dimensional Structures

Several two-dimensional (plane) truss structures have been analyzed previously by many authors. With reference to a fixed two-dimensional coordinate system (x_1,x_2) these include the (0°, 90°), (0°, 90°, ±45°) and (0°, ±60°) layups, as shown schematically in Figs. 2, 3, and 4, respectively. Here the 0° direction is assumed to coincide with the x_1 direction.

For the (0°, 90°) layup we identify two sets of parallel members with the direction cosines

$$(\beta_1,\beta_2)_1=(1,0) \qquad (\beta_1,\beta_2)_2=(0,1) \tag{12}$$

In Fig. 2 we kept the system general by allowing the two sets to have different properties and different spacings. From Fig. 2 we thus obtain:

$$Q_1^{(E)}=\frac{E_1A_1}{L_2h},\quad Q_1^{(T)}=\frac{E_1A_1\alpha_1}{L_2h},\quad Q_1^{(\rho)}=\frac{A_1\rho_1}{L_2h} \tag{13a}$$

$$Q_2^{(E)}=\frac{E_2A_2}{L_1h},\quad Q_2^{(T)}=\frac{E_2A_2\alpha_2}{L_1h},\quad Q_2^{(\rho)}=\frac{A_2\rho_2}{L_1h} \tag{13b}$$

where h is the constant height of the plate which is occupied by the rods.

Substituting from Eqs. (12) and (13a and b) into Eqs. (7-9) and summing, the results yield

$$C_{1111}=\frac{E_1A_1}{L_2h},\quad C_{2222}=\frac{E_2A_2}{L_1h},\quad C_{1122}=C_{2211}=0 \tag{14}$$

$$\gamma_{11}=\frac{E_1A_1\alpha_1}{L_2h},\quad \gamma_{22}=\frac{E_2A_2\alpha_2}{L_1h},\quad \gamma_{12}=\gamma_{21}=0 \tag{15}$$

$$\rho=\frac{\rho_1A_1}{L_2h}+\frac{\rho_2A_2}{L_1h} \tag{16}$$

For the (0°, 90°, ±45°) the geometry dictates that $L_1=L_2=L$; hence, we identify four sets of parallel members with the direction cosines

$$(1,0),\ (0,1),\ \left(\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),\ \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right) \tag{17}$$

From Fig. 3, we construct the Q's as

$$Q_1^{(E)}=\frac{E_1A_1}{Lh},\quad Q_1^{(T)}=\frac{E_1A_1\alpha_1}{Lh},\quad Q_1^{(\rho)}=\frac{A_1\rho_1}{Lh} \tag{18a}$$

$$Q_2^{(E)}=\frac{E_2A_2}{Lh},\quad Q_2^{(T)}=\frac{E_2A_2\alpha_2}{Lh},\quad Q_2^{(\rho)}=\frac{A_2\rho_2}{Lh} \tag{18b}$$

$$Q_d^{(E)}=\frac{E_dA_d}{\sqrt{2}Lh},\quad Q_d^{(T)}=\frac{E_dA_d\alpha_d}{\sqrt{2}Lh},\quad Q_d^{(\rho)}=\frac{A_d\rho_d}{\sqrt{2}Lh} \tag{18c}$$

Substituting from Eqs. (17) and (18) into Eqs. (7-9) and summing yields

$$C_{1111}=\frac{1}{Lh}\left(E_1A_1+\frac{E_dA_d}{2\sqrt{2}}\right),\quad C_{2222}=\frac{1}{Lh}\left(E_2A_2+\frac{E_dA_d}{2\sqrt{2}}\right) \tag{19a}$$

$$C_{1122}=C_{2211}=\frac{E_dA_d}{2\sqrt{2}Lh},\quad C_{1212}=\frac{E_dA_d}{2\sqrt{2}Lh} \tag{19b}$$

$$\gamma_{11}=\frac{1}{Lh}\left(E_1A_1\alpha_1+\frac{E_dA_d\alpha_d}{\sqrt{2}}\right) \tag{20a}$$

$$\gamma_{22}=\frac{1}{Lh}\left(E_2A_2\alpha_2+\frac{E_dA_d\alpha_d}{\sqrt{2}}\right) \tag{20b}$$

$$\gamma_{12}=\gamma_{21}=0 \tag{20c}$$

$$\rho=\frac{(\rho_1A_1+\rho_2A_2+\sqrt{2}\rho_dA_d)}{Lh} \tag{21}$$

For the (0°, ±60°) layup of Fig. 4 we shall assume that all members are identical and have the length L. Here again we identify three different sets of parallel members having the direction cosines

$$(1,0),\ \left(\frac{1}{2},\ \frac{\sqrt{3}}{2}\right),\ \left(\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right) \tag{22}$$

Moreover, the three sets will have the single mechanical, thermal, and density unidirectional properties

$$Q^{(E)}=\frac{2EA}{\sqrt{3}Lh},\quad Q^{(T)}=\frac{2EA\alpha}{\sqrt{3}Lh},\quad Q^{(\rho)}=\frac{2A\rho}{\sqrt{3}Lh} \tag{23}$$

Substituting from Eqs. (22) and (23) into Eqs. (7-9) finally yields

$$C_{1111}=\frac{3\sqrt{3}EA}{4Lh};\quad C_{2222}=\frac{3\sqrt{3}EA}{4Lh} \tag{24a}$$

$$C_{1122}=C_{2211}=\frac{\sqrt{3}EA}{4Lh};\quad C_{1212}=\frac{\sqrt{3}EA}{4Lh} \tag{24b}$$

$$\gamma_{11}=\frac{\sqrt{3}EA\alpha}{Lh};\quad \gamma_{22}=\frac{\sqrt{3}EA\alpha}{Lh} \tag{25a}$$

$$\gamma_{12}=\gamma_{21}=0 \tag{25b}$$

$$\rho=\frac{2\sqrt{3}\rho A}{Lh} \tag{26}$$

Three-Dimensional Structures

In our subsequent analysis we shall deal with two kinds of three-dimensional structures, namely "strictly three-dimensional" (see Figs. 5 and 9) and "quasi-three-dimensional" structures (see Figs. 6 and 10). The difference between these two kinds of structures is that the three dimensions of the strictly three-dimensional are comparable whereas two dimensions of the quasi-three-dimensional are much larger than the third. Thus, the quasi-three-dimensional model resembles a plate structure and hence may be referred to as two-dimensional. However, we prefer to refer to it as quasi-three-dimensional in order to avoid its confusion with the strictly two-dimensional (single layered) structures of Figs. 2-4.

In order to assess the utility of our analysis we shall derive the effective properties of two representative three-dimensional structures. We refer to the first as the octetruss (also referred to as tetrahedral[8]) and to the second as the cubic structures. Here, we indicate that we have previously analyzed the strictly three-dimensional version of these models by a somewhat similar but less direct approach. In what follows we shall find that the effective properties of the strictly three-dimensional models can be obtained by direct applications of Eqs. (9-11). The corresponding results for the quasi-three-dimensional model cannot be obtained directly but will be obtained from those of the strictly three-dimensional models by invoking the classical plate theory assumptions, i.e., the transverse normal stress σ_3 will be negligible compared to the remaining stresses.

Octetruss Model

The quasi-three-dimensional model of the octetruss structures consists of two parallel layers of 0°, ±60° rods

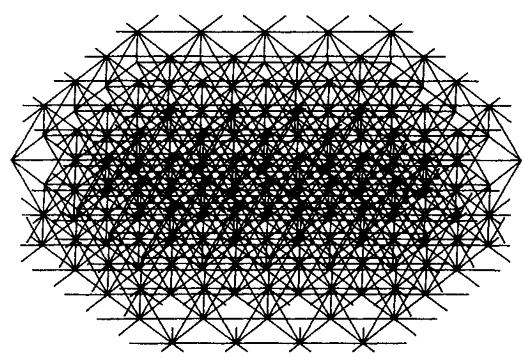

Fig. 5 Three-dimensional octetruss structure.

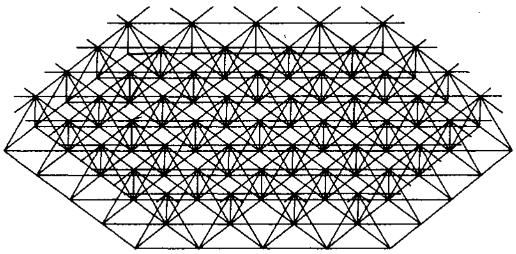

Fig. 6 Quasi-three-dimensional octetruss structure.

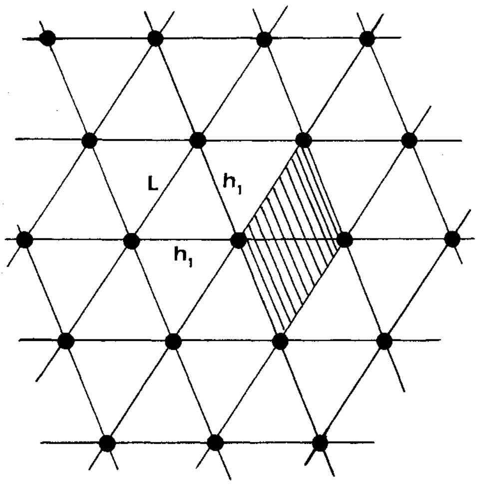

Fig. 7 The projected area of parallel-diagonal members in the octetruss model, $h_1 = (L/d)\sqrt{d^2-(L^2/4)}$.

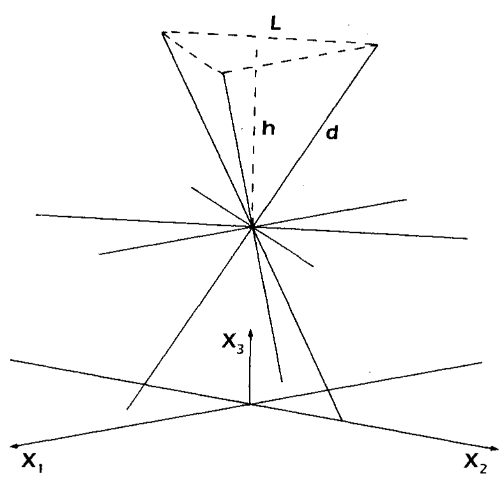

Fig. 8 Direction cosines of the octetruss.

connected by diagonal rods which form three-sided pyramids as shown in Fig. 6. In order to maintain generality we shall assume that the two layers have different properties from each other and also from the diagonals. We shall also assume that the two layers are an arbitrary distance h apart. Thus, if the rod length of the upper and lower surfaces is L then the length of each diagonal d will be

$$d = (h^2 + L^2/3)^{1/2} \tag{27}$$

This model has recently been analyzed by Noor et al.[8] using the energy method. Referring to Fig. 5 we see that the strictly three-dimensional octetruss can be obtained by stacking many of the quasi-three-dimensional layers at the top of each other.

Effective Unidirectional Properties of the Octetruss Model

Since the upper and lower layers of the repeating unit cell of the octetruss is made up of 0°, ±60° layups, their effective unidirectional properties can be obtained in a manner similar to that of the two-dimensional model of Fig. 4. If the upper and lower layers are designated by the subscripts 1 and 2, respectively, then by utilizing Eq. (23) we find

$$Q_i^{(E)} = \frac{2E_iA_i}{\sqrt{3}Lh}, \quad Q_i^{(T)} = \frac{2E_iA_i\alpha_i}{\sqrt{3}Lh} \tag{28a}$$

$$Q_i^{(\rho)} = \frac{2A_i\rho_i}{\sqrt{3}Lh}, \quad i = 1,2 \tag{28b}$$

Furthermore, the effective unidirectional properties of the diagonal rods can be easily obtained as follows. The projected area of a sub-octetruss structure consists of several repeating cells as shown in Fig. 7. On this figure, each heavy circular dot represents a column that is normal to the plane of the figure. By inspection, we see that each column occupies the effective area $\sqrt{3}\,hL^2/2d$. Thus by area averaging we find

$$Q_d^{(E)} = \frac{2dE_dA_d}{\sqrt{3}hL^2}, \quad Q_d^{(T)} = \frac{2dE_dA_d\alpha_d}{\sqrt{3}hL^2} \tag{29a}$$

$$Q_d^{(\rho)} = \frac{2dA_d\rho_d}{\sqrt{3}hL^2} \tag{29b}$$

Effective Properties of the Total Octetruss Structure

Referring to Fig. 5, we recognize that from each joint of the octetruss structure there emanate six double-lengthed columns. These are illustrated in Fig. 8.

With reference to the coordinate system of Fig. 8, the six member columns have the following direction cosines

$$(1,0,0), \ \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), \ \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right),$$
$$\left(\frac{L}{2d}, \frac{\sqrt{3}L}{6d}, \frac{h}{d}\right), \ \left(0, -\frac{L}{\sqrt{3}d}, \frac{h}{d}\right),$$
$$\left(-\frac{L}{2d}, \frac{\sqrt{3}L}{6d}, \frac{h}{d}\right) \tag{30}$$

If we substitute from Eq. (30) into Eq. (11) and use the appropriate unidirectional properties [Eqs. (28) and (29)] we finally obtain, for the strictly three-dimensional structures,

$$[C_{ijkl}] = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & 0 & 0 \\ C_{1122} & C_{1111} & C_{1133} & -C_{1123} & 0 & 0 \\ C_{1133} & C_{1133} & C_{3333} & 0 & 0 & 0 \\ C_{1123} & -C_{1123} & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{2323} & C_{1123} \\ 0 & 0 & 0 & 0 & C_{1123} & C_{1212} \end{bmatrix} \tag{31}$$

$$[\gamma_{ij}] = \begin{bmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{11} & 0 \\ 0 & 0 & \gamma_{33} \end{bmatrix} \tag{32}$$

and

$$\rho = \frac{2\sqrt{3}}{Lh}\left[(\rho_1A_1 + \rho_2A_2) + \frac{d}{L}\rho_dA_d\right] \tag{33}$$

where

$$C_{1111}=\frac{3\sqrt{3}}{4Lh}\left[(E_1A_1+E_2A_2)+\frac{1}{9}\frac{L^3}{d^3}E_dA_d\right] \quad (34a)$$

$$C_{1122}=\frac{\sqrt{3}}{4Lh}\left[(E_1A_1+E_2A_2)+\frac{1}{9}\frac{L^3}{d^3}E_dA_d\right] \quad (34b)$$

$$C_{1212}=\frac{\sqrt{3}}{4Lh}\left[(E_1A_1+E_2A_2)+\frac{1}{9}\frac{L^3}{d^3}E_dA_d\right] \quad (34c)$$

$$C_{1133}=\frac{1}{\sqrt{3}}\frac{h}{d^3}E_dA_d, \quad C_{1123}=\frac{1}{6}\frac{L}{d^3}E_dA_d \quad (34d)$$

$$C_{3333}=2\sqrt{3}\frac{h^3}{d^3L^2}E_dA_d, \quad C_{2323}=\frac{1}{\sqrt{3}}\frac{h}{d^3}E_dA_d \quad (34e)$$

$$\gamma_{11}=\frac{\sqrt{3}}{Lh}\left[(E_1A_1\alpha_1+E_2A_2\alpha_2)+\frac{1}{3}\frac{L}{d}E_dA_d\alpha_d\right] \quad (35a)$$

and

$$\gamma_{33}=2\sqrt{3}\frac{h}{dL^2}E_dA_d\alpha_d \quad (35b)$$

Notice that if we set $E_1A_1=E_2A_2=EA/2$ the results of Eq. (31) reduce identically to our previous results.[28] Notice also that $C_{1212}=(C_{1111}-C_{1122})/2$ and hence the octetruss is transversely isotropic.

Finally, we now specialize the results of the strictly three-dimensional octetruss to its quasi-three-dimensional model. This can be done by invoking the appropriate classical plate theory assumption, namely‡ $\sigma_{33}\approx 0$. With reference to Eq. (31) and to the format of Eq. (2′) we solve for ϵ_{33} as

$$\epsilon_{33}=-\frac{C_{1133}}{C_{3333}}(\epsilon_{11}+\epsilon_{22}) \quad (36)$$

This further restricts the results Eq. (31) which amounts to replacing C_{1111}, C_{1122} by $\bar{C}_{1111}$ and $\bar{C}_{1122}$, respectively, where

$$\bar{C}_{1111}=\frac{3\sqrt{3}}{4Lh}\left[(E_1A_1+E_2A_2)+\frac{1}{27}\frac{L^3}{d^3}E_dA_d\right] \quad (37a)$$

$$\bar{C}_{1122}=\frac{\sqrt{3}}{4Lh}\left[(E_1A_1+E_2A_2)-\frac{1}{9}\frac{L^3}{d^3}E_dA_d\right] \quad (37b)$$

Thus, the results of Eqs. (31), (32), and (33) [with the changes reflected as per Eq. (37)] define the complete thermo-mechanical properties of the quasi-three-dimensional octetruss. Except for the appearance of the multiplicative factor§ $1/h$ in our results and also except for some notational differences these results are identical with the zeroth moment of those derived by Noor et al.[8] Results corresponding to the first and second moment (the second moments, for example, lead to the definitions of plate-bending rigidities) can be easily obtained by multiplying the results of Eqs. (31-33) by z and z^2, respectively, and integrating across the plate's thickness $-h/2\leq z\leq h/2$. Since we are dealing with a discrete structure, contributions to the integrations are possible only when materials exist. Following Heki,[23] the value of the non-vanishing components of the first moments (designated as F) and the second moments (designated as D) of the present structure are obtained as:

$$F^{(E)}_{1111}=F^{(E)}_{2222}=\frac{3\sqrt{3}h}{8L}(E_1A_1-E_2A_2) \quad (38a)$$

$$F^{(E)}_{1122}=\frac{\sqrt{3}h}{8L}(E_1A_1-E_2A_2) \quad (38b)$$

$$F^{(E)}_{1212}=\frac{\sqrt{3}h}{8L}(E_1A_1-E_2A_2) \quad (38c)$$

$$F^{(T)}_{11}=F^{(T)}_{22}=(\sqrt{3}h/2L)(\alpha_1E_1A_1-\alpha_2E_2A_2) \quad (39)$$

$$F^{(\rho)}=(\sqrt{3}h/L)(\rho_1A_1-\rho_2A_2) \quad (40)$$

$$D^{(E)}_{1111}=D^{(E)}_{2222}=(3\sqrt{3}h^2/16L)(E_1A_1+E_2A_2) \quad (41a)$$

$$D^{(E)}_{1122}=(\sqrt{3}h^2/16L)(E_1A_1+E_2A_2) \quad (41b)$$

$$D^{(E)}_{1212}=(\sqrt{3}h^2/16L)(E_1A_1+E_2A_2) \quad (41c)$$

$$D^{(T)}_{11}=D^{(T)}_{22}=\frac{\sqrt{3}}{4}\frac{h^2}{L}(\alpha_1E_1A_1+\alpha_2E_2A_2) \quad (42)$$

and

$$D^{(\rho)}=\frac{\sqrt{3}}{2}\frac{h^2}{L}\left(\rho_1A_1+\rho_2A_2+\frac{1}{3}\frac{d}{L}\rho_dA_d\right) \quad (43)$$

which are identical to those reported in Ref. 8.

Cubic Model

Referring to Fig. 9 we see that the cubic structure consists of small cubes stacked in three dimensions. The repeating-unit cell consists of a single cube with member columns defining the outer edges, the diagonals, and the normals from the center of the cube to the respective outer surfaces. With the restriction that each of the outer edges is shared by four neighboring cubes (cells). It is obvious that two different column lengths are needed for its construction: those of the diagonals must be $\sqrt{3}L$ if the length of the edge columns is L. Here for simplicity we shall assume that all columns have the same properties and cross-sectional areas.

The quasi-three-dimensional version of the cubic structure is shown in Fig. 10. It consists of a single layer of unit cells (single cubed stacked in two dimensions). Accordingly, the strictly three-dimensional cubic material can be constructed by stacking many layers of the quasi-three-dimensional model.

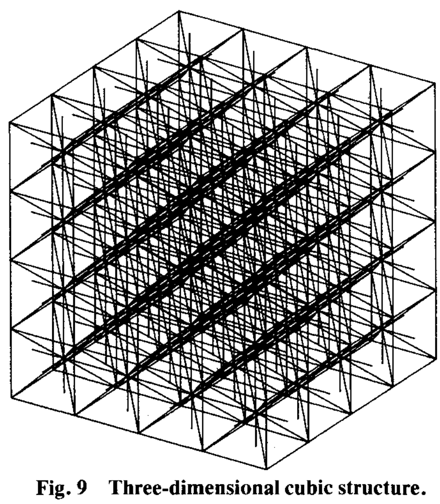

Fig. 9 Three-dimensional cubic structure.

‡This is also equivalent to the assumption of Eq. (22) of Ref. 8.

§The appearance of the multiplicative factor $1/h$ is due to the fact that we present our results in terms of stress-strain relations, whereas in Ref. 8, for example, results are presented in terms of force-strain relations.

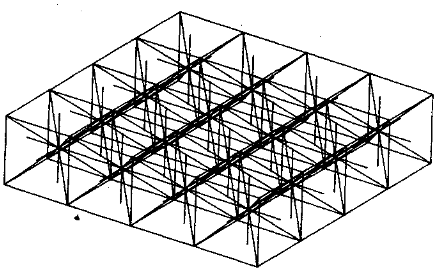

Fig. 10 Quasi-three-dimensional cubic structure.

Effective Unidirectional Properties of the Cubic Structures

Since all members of the cubic structure are assumed to have identical properties and cross-sectional areas we identify two unidirectional effective properties for each of the mechanical, thermal, and density properties. One of those, Q_d, corresponds to diagonal columns and the other, Q_l, corresponds to the rest. By inspection of Figs. 9 and 10 we can easily deduce these properties as (also see Ref. 28)

$$Q_l^{(E)} = \frac{2EA}{L^2}, \quad Q_l^{(T)} = \frac{2EA\alpha}{L^2}, \quad Q_l^{(\rho)} = \frac{2\rho A}{L^2} \tag{44a}$$

$$Q_d^{(E)} = \frac{\sqrt{3}EA}{L^2}, \quad Q_d^{(T)} = \frac{\sqrt{3}EA\alpha}{L^2}, \quad Q_d^{(\rho)} = \frac{\sqrt{3}\rho A}{L^2} \tag{44b}$$

Effective Properties of the Total Cubic Structure

Referring to Fig. 9 we recognize that from each joint of the cubic structure there emanate seven double-lengthed columns as is also illustrated in Fig. 11. From Fig. 11 we deduce that the seven member columns have the direction cosines

$$(1,0,0),\quad (0,1,0),\quad (0,0,1),\quad \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),$$
$$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \quad \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),$$
$$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \tag{45}$$

If we substitute from Eq. (45) into Eqs. (11) and use Eqs. (44), we finally obtain

$$C_{ijkl} = \frac{EA}{L^2}\begin{bmatrix} 2+\delta & \delta & \delta & 0 & 0 & 0 \\ \delta & 2+\delta & \delta & 0 & 0 & 0 \\ \delta & \delta & 2+\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta \end{bmatrix} \tag{46}$$

$$\gamma_{ij} = \frac{2\alpha EA}{L^2}\begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix} \tag{47}$$

$$\rho = 2\sqrt{3}\,\frac{\rho A}{L^2}\,(2+\sqrt{3}) \tag{48}$$

where

$$\delta = \frac{4}{3\sqrt{3}}, \quad \gamma = 1 + \frac{2}{\sqrt{3}} \tag{49}$$

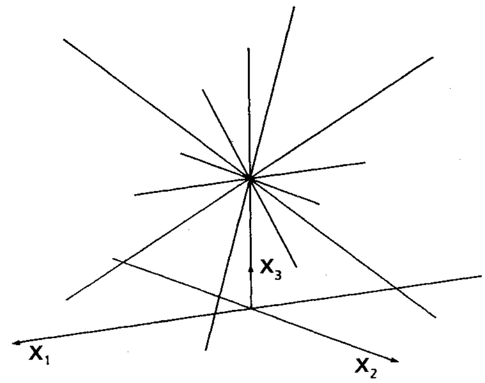

Fig. 11 Direction cosines of the cubic structure.

Results for the quasi-three-dimensional version of the cubic structure can be obtained by the same procedure outlined for the octetruss model. This leads to a modification of Eq. (46) in accordance with

$$C_{ijkl} = \frac{EA}{L^2}\begin{bmatrix} \frac{4(1+\delta)}{2+\delta} & \frac{2\delta}{2+\delta} & \delta & 0 & 0 & 0 \\ \frac{2\delta}{2+\delta} & \frac{4(1+\delta)}{2+\delta} & \delta & 0 & 0 & 0 \\ \delta & \delta & 2+\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta \end{bmatrix} \tag{50}$$

Thus, the results of Eqs. (47), (48), and (50) completely define the thermomechanical behavior of the quasi-three-dimensional cubic structure.

Finally, the results for the first and second moments of the quasi-cubic model can also be obtained by following the same procedure of the octetruss model. Due to the fact that the properties and cross-sectional areas of all members of the cubic structure are assumed to be the same, the components of the first moments, namely $F_{ijkl}^{(E)}$, $F_{ij}^{(T)}$, and $F^{(\rho)}$ vanish. On the other hand the nonvanishing components of the second moments are obtained

$$D_{1111}^{(E)} = D_{2222}^{(E)} = \frac{EA}{4}\,\frac{h^3}{L^2} \tag{51}$$

$$D_{11}^{(T)} = D_{22}^{(T)} = \frac{\alpha EA}{4}\,\frac{h^3}{L^2} \tag{52}$$

and

$$D^{(\rho)} = \frac{\rho A h^3}{3L^2}\,[2+\sqrt{3}] \tag{53}$$

V. Concluding Remarks

We have developed and utilized a simple and rather straightforward procedure in order to calculate effective mechanical, thermal, and density properties of large repetitive truss-like structures. Once the actual structure is specified, the construction procedure was outlined as follows: 1) all sets of parallel members are identified, 2) unidirectional "effective continuum" properties are derived for each of these sets, and 3) orthogonal transformations are finally utilized to determine the overall-effective properties of the structure.

Results are then specialized to two-dimensional and three-dimensional structures. In the case of two-dimensional structures three models were discussed. Those consisted of (0°, 90°), (0°, 90°, ±45°), and (0°, ±60°) layups with results displayed respectively in Eqs. (14-16), (19-21), and (24-26). Wherever available these results were found to be identical to those reported in the literature. In the case of three-dimensional structures we discussed two specific models. The first which we called strictly three-dimensional octetruss is a generalization of the two-dimensional (double-layered tetrahedral) model of Ref. 8, and the second we called cubic. In our analysis we referred to the double-layered tetrahedral model as a quasi-three-dimensional octetruss. Results for the quasi-three-dimensional models were obtained from those of the strictly three-dimensional model by invoking the classical-plate theory assumption that σ_3 be small compared with the remaining stresses. Our results for the quasi-three-dimensional octetruss were found to be identical with those reported by Noor et al.,[8] who used the energy equivalence approach.

We finally point out that using our procedure to derive effective properties of different quasi-three-dimensional models might lead to slightly different results from those which might be obtained using the energy equivalence method. These differences are perhaps a consequence of how one conceives of a continuum model to simulate actual structures.[33]

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